Extreme value Theory provides early warnings for critical transitions

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Early warnings of critical transitions have been extensively used to detect abrupt changes of dynamical regimes. In this paper we introduce new indicators based on the analysis of parametrically-modulated modifications in the properties of extremes for chaotic systems which possess experimentally accessible and detectable extreme value laws when far from bifurcation points. By measuring the deviation from the theoretically expected asymptotic distributions for long but finite samples, it is possible to detect the approaching critical transitions. Moreover, relations and connections between traditional and extreme value based indicators are explained and commented in detail. Numerical experiments have been performed on a stochastic differential equation describing the motion of a particle in an asymmetric double well potential under the effect of white noise. The results provide a gateway for using operatively the method described in other systems or data series analysis.

I. INTRODUCTION

An astonishing variety of complex systems ranging from finance to climate can experience abrupt changes at which a sudden shift of dynamical regime occurs. These so called “critical transitions” or “tipping points” can be formally seen as bifurcations in the dynamical systems terminology [1]. There is currently a great interest in understanding the dynamical behaviour in the proximity of a tipping point mainly for its importance in analysing and forecasting events of social and economic relevance [2, 3]. A growing interest in this topic has emerged and many indicators of criticality have been developed in order to identify early warnings of abrupt transitions to different dynamical states. The most used indicators are based on the modifications of the auto-correlation properties of particular observables when the system is pushed towards a transition, usually accompanied by an increase of the variance and the skewness of the distribution of the observable analysed [2–4]. Although for specific systems such techniques successfully highlight approaching tipping points, several difficulties prevent to extend these results to general cases. In particular, for high dimensional and complex systems featuring oscillations and intricate bifurcation patterns, early warning signals may be misleading. The main issue for these systems concerns the accessibility of dynamical and geometrical properties of the physical measure (i.e. the attractor) in the proximity of tipping points and the difficulties in recognizing the nature of the bifurcation involved. Moreover, since all the early warnings indicators rely on the knowledge of asymptotical statistical properties of the systems, so called “false alarms” - wrongly identified bifurcations - may arise when considering finite dataset and model outputs [3].

In this context, it is highly desirable to build early warnings indicators which provide themselves some estimations of dynamical and geometrical properties of the system so that false alarms can be discriminated from real early warnings. The main achievement of this paper is to introduce a new indicator of criticality based on the extreme value analysis which features the possibility to explore the asymptotic properties of the system and connect the fluctuations of maxima and minima of physical observables to the ongoing critical transition. To this purpose we exploit the theoretical results obtained so far for extreme values of dynamical systems to study the behaviour of a prototypical stochastic model in the proximity of a tipping point, showing analytically and numerically the capabilities of the new indicators.

Although our aim is to provide a general tool to analyse tipping points, the motivation for the research presented here originates mainly from the analysis of the data series of turbulent energy in the so-called plane Couette flow, i.e. the flow of a viscous fluid between two parallel plates, one of which is moving relative to the other [5]. In this system the control parameter is the Reynolds number whose value determines either turbulent or laminar flow regimes. We observed that, when the Reynolds number approaches the threshold at which the turbulence decays, the probability of observing a very low minimum of turbulent energy increases. As a consequence, maxima and minima exhibit an asymmetric behaviour that can be quantified by applying the tool provided by the Extreme Value Theory (EVT). This experimental result has been the starting point to devise a heuristic approach for more general scenarios, which is the subject of this paper. The results on the ongoing analysis of the breakdown of turbulence in the plane Couette flow, performed in collaboration with P. Manneville, will be reported in forthcoming papers.

The distribution of the largest or smallest values of certain observables has been widely studied as it is of great interest in many practical applications (e.g. the analysis of environmental extreme events). This interest has led

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to a fast development of a so-called extreme events theory [6–9]. The traditional approach introduced by Gnedenko [6] is based on the analysis of extremes obtained dividing the considered dataset into bins of fixed length and choosing the maximum for each bin. There, statistical inference is performed on such a reduced dataset by considering as model the Generalized Extreme Value (GEV) distribution family, which includes, as members, the Weibull, Gumbel and Frechet distributions, which greatly differ in terms of mathematical properties. Besides the block maxima approach, the so called Peak-over-Threshold (POT) is also widely used to tackle the problem of extremes. Extreme values are selected as exceedances over a certain threshold, fitted to the Generalized Pareto (GP) distribution. The asymptotic convergence to the GP model is then guaranteed by the Pickands-Balkema-de Haan theorem [7, 10].

Whereas the theory has been originally designed for the study of extremes for series of random variables, in the last decade the existence of asymptotic laws has been proven for maxima of observables computed on the orbits of dynamical systems. From the first rigourous paper by Collet [11], in the period of a few years relevant new results have been obtained [12–17]. The starting point of all these investigations has been associating to the stationary stochastic process given by the dynamical system a new stationary independent sequence distributed according to the GEV model, and then pulling back the obtained statistical laws to the original sequence extracted from the dynamical system. The assumptions necessary to observe a GEV distribution in dynamical systems rely on the choice of suitable observables and the fulfilment of particular mixing conditions. These results can also be used to study extremes of stochastically perturbed dynamical system as recently shown in [18, 19].

In order to construct robust indicators of criticality, able to discern when the underlying system is close to a bifurcation, we will take two complementary points of view:

1. The classical approach of looking at the autocorrelation properties of the time series of the considered variable is replaced with the investigation of the appropriateness of GEV for the extremes of the time series: the presence of long time-correlations is in antithesis with the fulfilment of the mixing conditions referred to above.

2. The search for changes in the skewness of the bulk statistics is substituted with the analysis of whether parametric modulations to the system lead to changes in the qualitative properties of the extremes, i.e. whether we observe transitions among the three kinds of distributions included in the GEV family.

The capabilities of these new indicators will be analysed for probably the simplest conceivable dynamical model featuring multi-stability, i.e. the overdamped 1D motion of a stochastically forced particle in a double well potential. The choice of this model naturally relies on the fact that, even for complex and chaotic dynamical systems which feature tipping points, a reduction to the Langevin model is often attempted as it opens the way to approaching the problem in terms of the one-dimensional Fokker-Planck equation [20], even if one must handle with care the construction of a surrogate, stochastic dynamics, as discussed in [21] regarding a relevant example of geophysical relevance. Moreover, this model can be easily explored numerically and analytically. In future papers we will test the indicators in complex systems arising from fluid dynamics.

The fundamental reason why we believe that EVT-based indicators are preferable to usual methods based on the bulk of the distribution lies on the fact that, whereas there is, obviously, no universal model for the bulk statistical properties of a given system, such universality, even if with limitations, exists exactly for extremes for which EVT provides a universal family of parametric distributions.

This work is organised as follows: in Section 2 we present some basic notions of extreme value theory for independent and identically distributed variables and dynamical systems. In Section 3 we introduce the new indicators and describe their properties. Section 4 is devoted to explaining the theoretical and numerical results for the motion of a particle in an asymmetric double well potential under the effect of noise. Finally in Section 5 we review the results obtained for the system analysed and propose an algorithm to extend the analysis to other dynamical systems or time series drawing our conclusions and more suggestions for further work.

II. ELEMENTS OF EXTREME VALUE THEORY FOR DYNAMICAL SYSTEMS

A. Traditional Extreme Value Theory

Gnedenko [6] studied the convergence of maxima of i.i.d. variables

$$X_0, X_1, ..X_{m-1}$$

with cumulative distribution (cdf) of the form:

$$F_m(x) = P\{a_m(M_m - b_m) \leq x\} \tag{1}$$

Where a_m and b_m are normalizing sequences and $M_m = \max\{X_0, X_1, ..., X_{m-1}\}$. Under general hypothesis on the nature of the parent distribution of data, it has been shown that the distribution of maxima, up to an affine change of variables, converges in the limit for $m \to \infty$ to a single family of generalized distribution called GEV distribution with probability density function:

$$h_G(x;\mu,\alpha,\kappa) = \frac{1}{\alpha}\, t(x)^{\kappa+1} e^{-t(x)} \quad (2)$$

where

$$t(x) = \begin{cases} \left(1+\kappa(\frac{x-\mu}{\alpha})\right)^{-1/\kappa} & \text{if } \kappa \neq 0 \\ e^{-(x-\mu)/\alpha} & \text{if } \kappa = 0 \end{cases} \quad (3)$$

which holds for $1+\kappa(x-\mu)/\alpha > 0$, using $\mu \in \mathbb{R}$ (location parameter) and $\alpha > 0$ (scale parameter) as scaling constants in place of b_m, and a_m [7, 19]. $\kappa \in \mathbb{R}$ is the shape parameter, also called the tail index, and discriminate among the classical Extreme Value Laws (EVLs): when $\kappa \to 0$, the distribution corresponds to a Gumbel type (type 1 distribution). When the index is positive, it corresponds to a Fréchet (type 2 distribution); when the index is negative, it corresponds to a Weibull (type 3 distribution). The type of distribution observed is very important as it discriminates the kind of tail decay of the parent distribution. A Gumbel distribution is typically observed when the parent distribution features an exponential tail, whereas Fréchet and Weibull laws are typical associated to power law tails: a Weibull is usually observed when the tail is limited above by a certain threshold whereas a Fréchet when the tail is limited below. In order to compare the properties of maxima and minima distributions, they should both be treated as they correspond to right or left tails of the parent distribution. In the forthcoming analysis, we will always change sign to the minima distribution [22].
As shown in Eq. 1, Gnedenko's results are related to a precise way of selecting extremes, the so called block maxima approach: it consists of dividing the data series of length s of some observable into n bins each containing the same number m of observations, and selecting the maximum (or the minimum) value in each of them [22]. The extremes are then fitted to the GEV distribution. In Section 4.A we will give a detailed description of the inference method used to extract the parameters of the EVLs.

B. Extremes of Dynamical Observables

We now consider the generalization of EVT to dynamical systems. The results presented in this section are provided in terms of maps (discrete-time dynamical systems) but they are also valid for flows (continuous-time dynamical systems).
Let us consider a dynamical system $(\Omega, \mathcal{B}, \nu, f)$, where Ω is the invariant set in some manifold, usually $\mathbb{R}^d$, $\mathcal{B}$ is the Borel σ-algebra, $f: \Omega \to \Omega$ is a measurable map and ν a probability f-invariant Borel measure.
In order to adapt the extreme value theory to dynamical systems, we will consider the process $X_0, X_1, ...$ given by:

$$X_t(x) = g(\mathrm{dist}(f^t(x), \zeta)) \qquad \forall t \in \mathbb{N} \quad (4)$$

where 'dist' is a distance on the ambient space Ω, ζ is a given point and g is an observable function. As we said above, we will use three kinds of observables $g_i, i = 1, 2, 3$, defined as:

$$g_1(x) = -\log(r) \quad g_2(x) = r^{-\beta} \quad g_3(x) = C - r^{\beta} \quad (5)$$

where $r = \mathrm{dist}(x, \zeta)$, C is a constant and $\beta > 0 \in \mathbb{R}$. Using these observables we can obtain convergence to a distribution of type 1,2,3 if one can prove two sufficient conditions called D_2 and D' that the dynamical system obeys. These conditions require the presence of a sufficiently fast decay of correlation for the stochastic dynamical sequence and limit the possibility of having clustered extremes. Another way to approach the problem of extremes for the g_i observables relies on studying the statistics of first return and hitting times, which provide information on how fast the point starting from a certain initial conditions returns in its neighbourhood, see the papers by [13] and [23]. They showed in particular that for dynamical systems preserving an absolutely continuous invariant measure or a singular continuous invariant measure ν, the existence of an exponential hitting time statistics on balls around ν almost any point ζ implies the existence of extreme value laws for one of the observables of type $g_i, i = 1, 2, 3$ described above. The converse is also true, namely if we have an extreme value law which applies to the observables of type $g_i, i = 1, 2, 3$ achieving a maximum at ζ, then we have exponential hitting time statistics to balls with center ζ. Recently these results have been generalized to local returns around balls centred at periodic points [15].
Since it is difficult to check the exponential decay of hitting time statistics as recurrences are hard to tackle analytically or numerically in more than three dimensions, it is highly desirable to connect the theory explained above to a more straightforward dynamical indicator computable for a wide class of observables as well as in high dimensional dynamical systems. On the other hand, condition D_2, introduced in its actual form by Freitas-Freitas [12], could be checked directly by estimating the rate of decay of correlations for Hölder observables. Starting from this observation, Aytaç et al. proved the existence of EVLs for dynamical systems whose correlations decay is at least summable [18]. Since these decays can be easily computed numerically, in the following discussion, instead of checking on the mixing conditions D_2 and D', we will refer exclusively to the results described in [18].
In [24] and [25] the authors have verified the convergence to the classical EVL and the relations with the GEV family from both an analytical and numerical point of view in a wide class of mixing low dimensional maps, showing that - in mixing maps - GEV shape parameters only depend on the local dimension of the attractor $\delta(\zeta)$:

$$\kappa(g_1) = 0 \qquad \kappa(g_2) = \frac{\beta}{\delta(\zeta)} \qquad \kappa(g_3) = -\frac{\beta}{\delta(\zeta)} \qquad (6)$$

In [26] it is shown that a GEV distribution can be fitted only if the mixing conditions are fulfilled whereas other kind of distributions not belonging to the GEV family are observed for quasi-periodic and periodic motions. The theory has been extended to stochastically perturbed dynamical systems in [18]: in particular, systems whose trajectories under the deterministic dynamics evolve towards a finite number of fixed points possess EVLs when perturbed with additive noise. In these cases, the extreme value parameters depend on the phase space dimension and not on the local dimension as the noise helps the perturbed system to explore the neighbour of a fixed point in a ball which scales exactly with the dimension of the phase space. In [19], the authors define some guidelines to effectively observe such EVLs in numerical simulations, pointing out the range of noise to be used. These results will be particularly useful for describing the behaviour of the stochastic differential equation whose deterministic part is driven by a double well potential dynamics which will be extensively analysed in Section 4.

In the next section we will describe what we expect to happen in a general case. The notations will refer to a deterministic dynamical system which features at least two disjoint attractors but - under the conditions previously described - the results are valid also in the stochastically perturbed case.

III. TIPPING POINTS AND EXTREMES

Let us assume that ν is the physical measure of $f : \Omega \to \Omega$. In fact, since a dynamical systems may have several different invariant measures, we will always refer to the physical measure as the one arising naturally in any numerical simulations. We will also assume that for a given initial condition x_0, the physical measure $\nu(x_0)$ is unique but it is not the same for all the initial conditions in the phase space, i.e. there are at least two disjoint attractors. Let us perturb the dynamics by introducing a parameter λ such that $f = f(x_0, \lambda)$ and $\nu = \nu(x_0, \lambda)$. We define λ_c as the critical value of λ such that for $0 \leq \lambda < \lambda_c$ all the disjoint attractors continue to exist - each one with its own basin of attraction - but, whenever $\lambda \geq \lambda_c$, the system undergoes a bifurcation that makes one of the disjoint attractors disappear. For $f(\lambda < \lambda_c)$, the system has a summable decay of correlations and therefore the block maxima of the observables of the form $g_i(\cdot)$ asymptotically obeys one of the EVLs described above. We have already mentioned in the introduction that the behaviour of the system changes in the limit $\lambda \to \lambda_c$ as the correlations in the system increases [2, 27]. Therefore, if the decay of correlations gets slower and slower the order of extreme statistics - in practise, the minimum length of our experimental bin from which we extract the maximum - needed to observe convergence to the predicted EVLs becomes higher and higher. In the limit, the bin length needed to decorrelate the data tends to infinity and this prevents from obtaining EVLs. At finite time, this effect gives rise to increasing deviations from the theoretical behaviour that can be explored numerically whenever the asymptotic expected values for μ, α and κ are known. This condition is easily met when dealing with systems whose physical measure is absolutely continuous with respect to Lebesgue. In this case the EVL parameters given in Eq. 6 only depend on the phase space dimension. For stochastically perturbed dynamical systems this is also true provided that we choose a suitable noise amplitude as explained in [19]. Moreover, once found the bin length needed to obtain the asymptotic results for λ such that in the perturbed system we do not observe any critical transition, we can fix it to study the convergence to EVLs when approaching λ_c.

In high dimensional systems, the theoretical framework described above is still valid even though it may be difficult to perform numerical simulations and control the variables in the phase space. In this case, critical transitions are highlighted by specific physical observables that experience abrupt changes when the system crosses a tipping point. These observables undergo greater amplitude deviations in the direction of the state they are destined to shift to, as resulting from the bifurcation, than in the opposite direction, often showing an increase in the skewness of the distribution. Recently, the skewness increase near a tipping point has been proposed as an early warning indicator by Guttal et al. [28]. However, the method does not allow for having a quantitative estimation of the threshold value to be recognized as λ_c. An extreme value analysis on the maxima and on the minima of the distribution is capable of overcoming this problem and highlight the critical transition by providing a quantitative way to determine the value of λ_c. Let us call the physical observable ϕ, for $\lambda < \lambda_c$, the extreme fluctuations of the observable are confined in the direction of the minima and of the maxima giving rise to a Weibull distribution typically observed for series bounded by an upper threshold. Instead, when $\lambda \to \lambda_c$ the fluctuations feel the presence of the other attractor in the direction of the state it is destined to shift to and this is reflected by a change in the extreme distributions from a Weibull one to a Gumbel one. This mechanism also allow to understand in which direction the transition will take place if the goal is to use the early warnings in a predictive way.

The critical indicators introduced so far using the EVT results can be divided into two classes: *dynamical indicators* based on the divergence of the classical EVLs in the proximity of tipping points for the maxima of $g_i \quad i = 1, 2, 3$ and *physical indicators* based on the

changes in the distributions of maxima ($\max(\phi)$) and "reversed minima" ($-\min(\phi)$) of a certain relevant physical observable ϕ near the critical transitions. In the next section we will show an example in which the behaviour for the critical indicators introduced in this section can be derived analytically and observed numerically, defining guidelines for the application in other systems in Section 5.

IV. A CASE STUDY

The system considered in the following analysis is widely used for modelling several natural phenomena featuring a bistable behaviour.
The following stochastic differential equation (SDE):

$$dx = -V'(x)dt + \epsilon dW, \tag{7}$$

describes the over-damped motion of a material point in a potential under the effect of a stochastic forcing. The potential can be written as $V(x) = \frac{1}{4}x^4 - ax^2 + cx$ with $a, c > 0$ and it is represented in Fig. 1 for some values of the parameters a and c. The stochastic forcing consists of a Wiener process W whose amplitude is modulated by the parameter $\epsilon > 0$. By letting $\epsilon = 0, c = 0$ the system is completely deterministic and features two stable fixed points ($\bar{x_1} < 0$ and $\bar{x_2} > 0$) and one unstable fixed point. It is trivial to check that extremes of any observables extracted from the deterministic dynamics do not obey any of the classical EVLs. Instead, as soon as noise is switched on by setting a non-zero stochastic forcing, the physical measure becomes absolutely continuous with respect to Lebesgue so that, starting from any initial conditions x_0, asymptotic EVL exists at any point ζ. In fact, the probability distribution for the variable x can be computed by solving the Fokker Planck equation and considering a steady solution [29]. The result is obeyed in general and it is known as the Boltzman factor:

$$\rho^\epsilon(x) = C(\epsilon)\exp\left\{-\frac{2V(x)}{\epsilon}\right\} \tag{8}$$

The dynamical behaviour of Eq. 7 can be understood by computing the mean exit time for the basin of attraction of $\bar{x_2} > 0$ as in [30], which for small values of c can be written as:

$$\langle\tau(\bar{x_2})\rangle = \frac{\pi}{2^{3/2}a}\exp\left(\frac{2a^2}{\epsilon^2}\left(1 - \frac{4c}{(2a)^{3/2}}\right)\right). \tag{9}$$

while the mean exit time from the basin of attraction of x_1 is obtained by changing c to $-c$.
For $c = 0$ the mean exit time from the negative and positive basin are the same. Instead for $c > 0$, fixing the initial condition in $x_0 = \bar{x_2}$, we notice that the mean exit time decreases when $\frac{4c}{(2a)^{3/2}} \to 1$. Once ϵ is fixed such that $0 < \epsilon \ll a$, the average exit time can be controlled by modulating c.

Let us now analyse the behaviour of the *dynamical indicators* described in the previous section when c is changed. When noise amplitude is small and a local equilibrium approximation is used, the indicators converge towards the EVL as described in Eq. 6 with $\delta(\zeta) = 1$ for all ζ since it has been proven for the double well symmetric potential that the correlation decay is a stretched exponential (see Eqs. 47,48 in [31]). As pointed out in the previous section, such a decay of correlations is enough to prove that g_i extremes have classical EVLs. Instead, when $\langle\tau(\bar{x_2})\rangle \to 0$, the correlation decay becomes slower and slower (compare Fig. 4 and Fig. 5 in [31]) and the extremes will still obey a classical EVL but it becomes practically impossible to observe in a finite time series: this points at the importance of looking not only at the asymptotic statistics of processes, but also at the rate of convergence.

As *physical indicator* we simply examine the maxima and reversed minima distribution of $\phi(x) = x$ which here represents the position of the particle in the right well of the potential $V(X)$ being the initial condition $x_0 = \bar{x_2}$. We remind that in a general case the analysis can be carried out by analysing the series of any relevant observables of the systems whose expectation value is different in the two basins of attraction. When $c \to 0, \epsilon \ll a$, an harmonic expansion of the potential around $\bar{x_2}$ holds and the particle is confined in both the direction of the minima and the maxima. In terms of extremes distributions we observe in both cases a Weibull type for the reasons described in the previous section. For increasing c the harmonic approximation of the potential does not hold any more and we have to consider the full expression of $V(x)$: the maxima are still bounded by the quartic term but, towards the minima, the particle "feels" the presence of the other well experiencing a change of the distribution from Weibull to a Gumbel type.

A. Numerical experiments

The numerical experiments, performed on the one dimensional SDE described in Eq. 7, are devised to follow exactly the theoretical set-up described in this section. An ensemble of 100 realisations of the system is produced starting from $x_0 = \bar{x_2}(c = 0)$, the noise amplitude is set at $\epsilon = 0.5$, $dt = 0.1$ under the potential $V(x) = 1/4x^4 - 1/2x^2 + cx$ represented in Fig. 1. For each realisation, $n = 1000$ extremes are selected with two different bin length: $m = 1000, m = 2000$ for a total number of iterations $s = 10^6$ and $s = 2 \cdot 10^6$ respectively. Even though for $m = 1000$, at $c = 0$, we get the shape

parameters as predicted from the theory, we repeat the experiment for increased s in order to show that the divergence at c_{crit} is not removed by considering longer time series. Approaching the critical transition, whenever the particle falls in the left well we interrupt the realisation reinitialising the initial condition x_0 counting, among all the realisations, the number of transitions experienced, which provides an indication on how likely a transition will occur during the time interval considered. Even if we expect our indicators to be effective as soon as transitions are observed, the reinitialisation helps in studying the behaviour of the indicators in the interval of c values for which the jump to the other basin of attraction is probable but not certain. This may help whenever only a realisation of the system is considered or when dealing with experimental data set. The inference procedure follows [32, 33] where the authors have used the L-moments estimation described in[34].

Let us first analyse what happens for the *dynamical indicators* for which results are shown in Fig. 2 where the shape parameters κ for the g_i observable are displayed against the values of c. The left plots correspond to the case $m = 10^3$, $\beta = 1/3$, the right ones to the case $m = 10^4$, $\beta = 1/3$. Substituting the value of the parameters chosen for the simulation in Eq.9, the exponential term vanishes when $c \to 0.25 = c_{crit}$. Nonetheless, the value of c for which the system experiences at least one transition from one basin of attraction to another and a reinitialisation from the initial condition is needed is about $c_{crit} \simeq 0.21$ in both the experiments. The difference between the theoretical and experimental value of c_{crit} is due to the fact that the noise has a finite size.
The divergence is far more evident for the observable g_1 and g_3 whereas it seems slower for the g_2. This is due to the different weight assigned by g_2 to the minimum distances: in fact, with respect to the observables g_1 and g_3, g_2 weights more the points which come closer to ζ. Since spurious extremes are located relatively far away from ζ, they do not contribute sensibly to the divergence of the shape parameter of g_2. This asymmetry with respect to the other observables may be removed by choosing a smaller β so that farther extremes are weighted more. Repeating the experiment setting $\beta = 1/10$ for g_2 we obtain an estimation of c_{crit} consistent with the other observables. To support this claim we have carried out the same experiments - not shown here - for the potential $V(x) = ax^2$ with $a > 0$, computing the statistics with respect to the point $\zeta = 0$. In this case, by setting $a \to 0$, the potential gets flatter so that the material point can explore a wider region of the phase space causing the appearance of spurious maxima in the distribution for finite size of m. This is reflected in the behaviour of g_2 which follows exactly the case of the double well potential so that for $\beta = 1/3$ the shape parameters diverges slower from the expected value than in the case $\beta = 1/10$.
The divergent behaviour appears at lower values of c in the case $m = 1000$ with respect to the other case. In fact, by increasing the bin length the problem of lower decay of correlations appears at higher c than in the case $m = 1000$ as we improve the convergence by selecting more "authentic" extremes. Encouragingly, also in the $m = 2000$ experiment, the critical transition can be highlighted well before the probability of jumping to the other attracting states is higher than the 20%. From this example it is clear that is impossible to univocally define a threshold that indicates the tipping point. This is an intrinsic problem of the critical transitions in systems driven by noise and it is directly related to the time scales on which we observe our system. Therefore the divergence obtained by the use of different g_i should be critically analysed to detect the approaching tipping point. Instead, in a modelling framework, one can test the system's behaviour with changing levels of noise. As discussed in [21] when looking at average transitions rates, in a non one dimensional system, one should expect the same scaling properties of c_{crit} and ϵ is the same as in the simple one dimensional model analysed here.
The results for the *physical indicators* are shown in Fig.3. The upper plots represent the shape parameter for the maxima of $\phi(x) = x$, the middle plot the same but for the reversed minima whereas the lower plot is the same of Fig. 1 repeated here for convenience. The set-up is exactly the same as for the *dynamical indicators*. In this case what we observe is a change in the type of distribution only in the direction of the minima. This is explainable by observing that an increase the bin length, the probability of observing a very low minimum within a single bin increases and this reflects in the overall distribution. In the case $m = 2000$ we observe even a Fréchet law but for the values of the system for which the system has experienced a critical shift at least in the 20% of the realisations. In this case it seems indeed possible to define c_{crit} as the value for which the GEV distribution of the minima or of the maxima changes sign. For more CPU demanding experiments or for experimental datasets an ensemble of realisations is usually not available. In these cases the switching between different types of distribution for only the minima - or the maxima - may be still interpreted as a signal of an approaching tipping point using confidence intervals as uncertainty for the parameters instead of the standard deviation of the sample. This approach we also applied to the analysis of turbulent energy data in the plane Couette flow about which we will report elsewhere.

V. FINAL REMARKS

This paper has addressed the problem of using extremes statistics to define robust indicators for approaching tipping points of dynamical systems. The indicators have been grouped into two categories: *dynamical indicators* and *physical indicators*. For the former, by knowing some properties of the physical

measure it has been possible to identify the asymptotic EVLs when the system is well away from bifurcation points and thereby detect approaching tipping points by the divergence from the expected EVLs. Similarly, for the *physical indicators* constructed using extremes of relevant observables of the system, a comparison between maxima and reversed minima asymptotic EVL parameters provide a straightforward way to identify critical transitions. From the numerical results, described in the previous section, we suggest some guidelines to implement the indicators presented in an algorithmic way as described in Fig.4:

- The *dynamical indicators* are mainly devised for applications in low dimensional systems as the trajectories must be explicitly computed. For deterministic dynamical systems the local dimension should be known at the point ζ whereas for stochastic dynamical systems the asymptotic EVLs depend only on the phase space dimension. The algorithm, represented schematically in the upper panel of Fig. 4, begins by setting $\lambda = 0$ and the computation of the length of the series s needed to obtain theoretical parameters consistent with the asymptotic EVLs. Once s is determined, λ is increased and the fitting procedure repeated until the GEV parameters diverge from the theoretical expected one. The critical λ may be recognized when the experimental parameters are not consistent with the theoretical ones.

- The *physical indicators* algorithm is described in the bottom panel of figure 4. It can be applied to a series of observables $\phi(x)$ originating from dynamical systems or to an experimental dataset. The first step, as in the previous case, is to chose a suitable s in order to obtain a Weibull law for maxima and reversed minima of $\phi(x)$ in the unperturbed system. Once the length of the data series s is fixed, the critical λ is the one for which the shape parameter of the distribution either of the minima or of the maxima changes sign - provided that the other remains negative.

The extreme values indicators present some noticeable advantages with respect to the indicators commonly used to highlight critical transitions: the knowledge of the asymptotic EVLs provides itself a robust indication of the order statistics needed to profit from an extreme value analysis. In other terms, we have the exact statistical model to conform to (under given conditions on the dynamical system's properties), and we can correctly infer that if the statistics does not obey the GEV model, then the conditions are not obeyed. This gives a much stronger mathematical framework to our analysis than in most previous investigations. In many applications this information is not available and this causes the identification of early warning signal as false tipping points. For the *physical indicators*, the interesting intuition that the skewness of the distribution changes when approaching the transition [28] can be related to modifications in the EVLs quantifiable with a change of sign of the shape parameters corresponding to a different type of extreme value distribution. Clearly, results will improve by choosing observables whose values change more in percentage crossing the tipping point. The strict connection between the *dynamical indicators* and the recurrences in a point suggest that we are able to highlight critical transitions only by looking at the behaviour of the systems in the neighbour of a specific point ζ which can be located in any point of the attractor. This could help in all the situations for which the dynamic may be better represented or analysed in a sub-domain of the phase space. Moreover these indicators themselves provide interesting information on events which happen with very small probability but that are usually relevant in ecosystems, climate and financial models. Certainly the methods based on extreme value statistics require a great availability of good quality data and/or the possibility to perform time consuming simulations capable to extract authentic extremes. This problem is especially relevant for applications in climate science and in finance since the models involved are very demanding in terms of CPU time and long-term data availability is poor. However, such data will be more and more available in the future and not only for simplified models.

The numerical tests carried out demonstrate the applicability of the proposed indicators in practical applications. Although the results meet the theoretical setup and provide warning signals for approaching tipping points, it is indeed evident that other tests should be carried out to assess the general validity of this approach. We have already successfully tested the indicators on bidimensional systems obtaining results comparable to the ones shown in this paper and that will be reported elsewhere. Since the statistical tools used to study extreme values are commonly distributed with scientific software, the algorithm can be easily checked and compared to other methods. Our aim is to test the indicator on complex systems arising from fluid dynamical studies which feature tipping points whose nature remain not well understood and on other theoretical low dimensional models. One must bear in mind that, as discussed in [21], great caution must be paid when trying to extend considerations valid for one dimensional models onto higher dimensional systems: the operation of defining a one dimensional effective projected dynamics is far from being trivial.

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We regret that on October 22nd 2012 an Italian court has sentenced seven scientists - Barberi, Boschi, Calvi, Dolce, De Bernardinis, Eva and Selvaggi - to six years in prison for, at all practical levels, not having been able to predict the L'Aquila earthquake in 2009. This sentence seems to have no basis whatsoever on the current scientific knowledge. Instead, we wish to acknowledge them for their efforts and contributions in the assessment of risk factors linked to geophysical phenomena.

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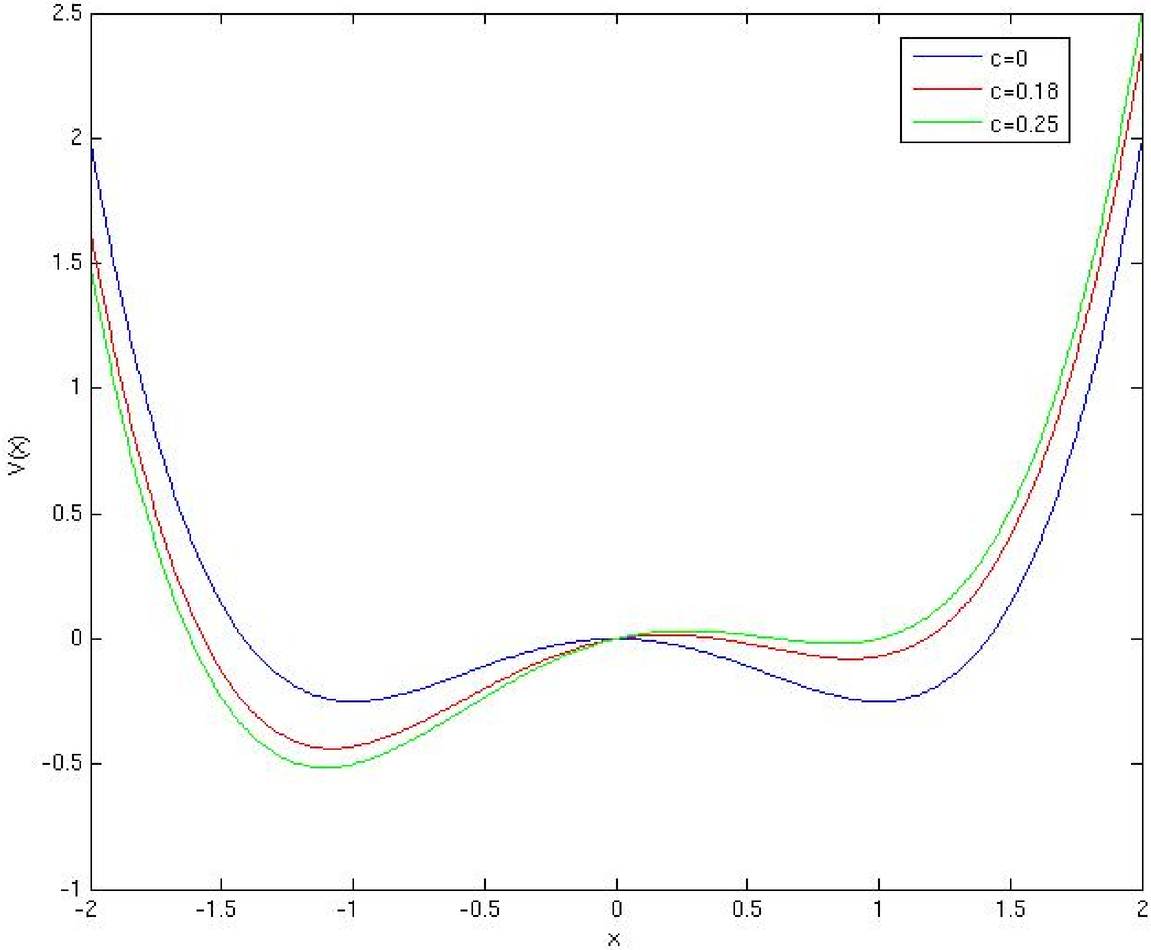

FIG. 1. Potential $V(x) = 1/4x^4 - 1/2x^2 + cx$ for different values of parameter c.

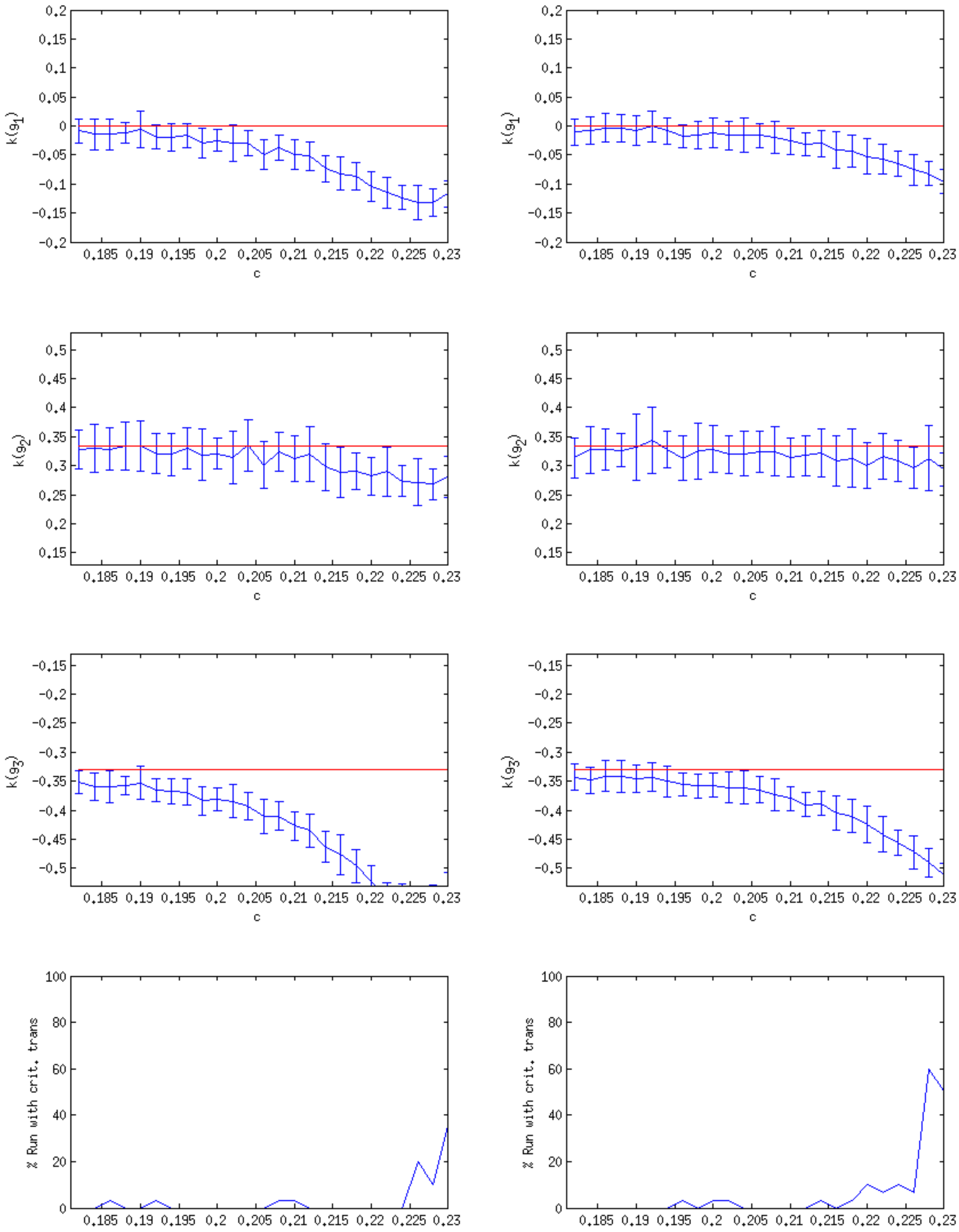

FIG. 2. Extreme value shape parameter κ vs the potential asymmetry parameter c for an ensemble of 30 trajectories starting in $x_0 = \bar{x_2}$ of the system described in Eq. 7. **(a)**: g_1 observable, **(b)**: g_2 observable with $\beta = 1/3$, **(c)**: g_3 observable with $\beta = 1/3, C = 1$, **(d)**: percentage of runs for which the particle jump at least once in the left% well. **Left**: $n = 1000$, $m = 1000$. **Right**: $n = 1000$, $m = 2000$. The errorbars represent a standard deviation of the sample, the red bars represent theoretical expected parameters for $c = 0$ case.

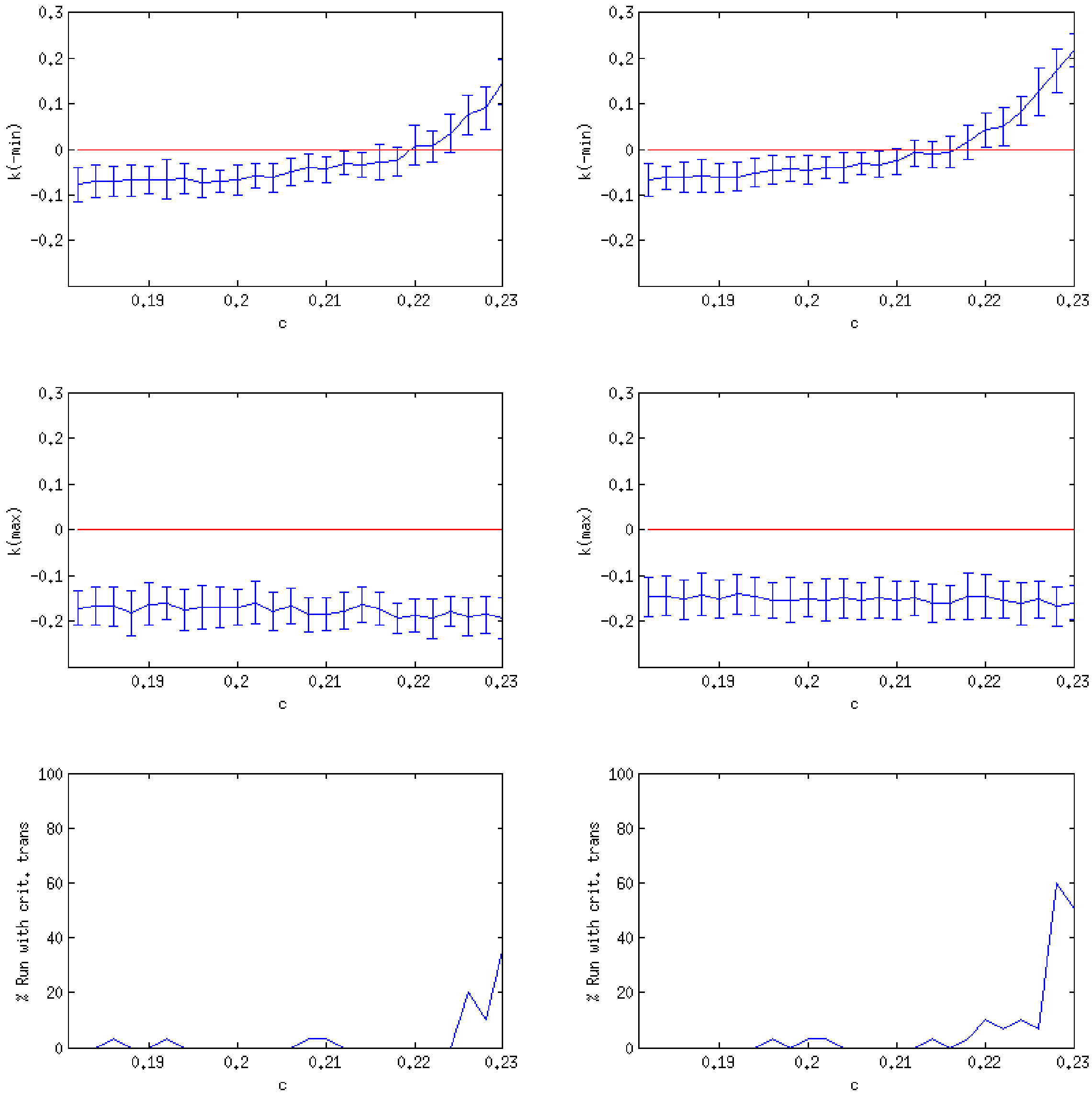

FIG. 3. Extreme value shape parameter κ vs the potential asymmetry parameter c for an ensemble of 30 trajectories starting in $x_0 = \bar{x_2}$ of the system described in Eq. 7. **(a)**: maxima for the observable $\phi(x) = x$ **(b)**: reversed minima for the observable $\phi(x) = x$; **(c)**: percentage of runs for which the particle jump at least once in the left% well. **Left**: $n = 1000$, $m = 1000$. **Right**: $n = 1000$, $m = 2000$. The errorbars represent a standard deviation of the sample, the red bars represent the Gumbel distribution ($\kappa = 0$).

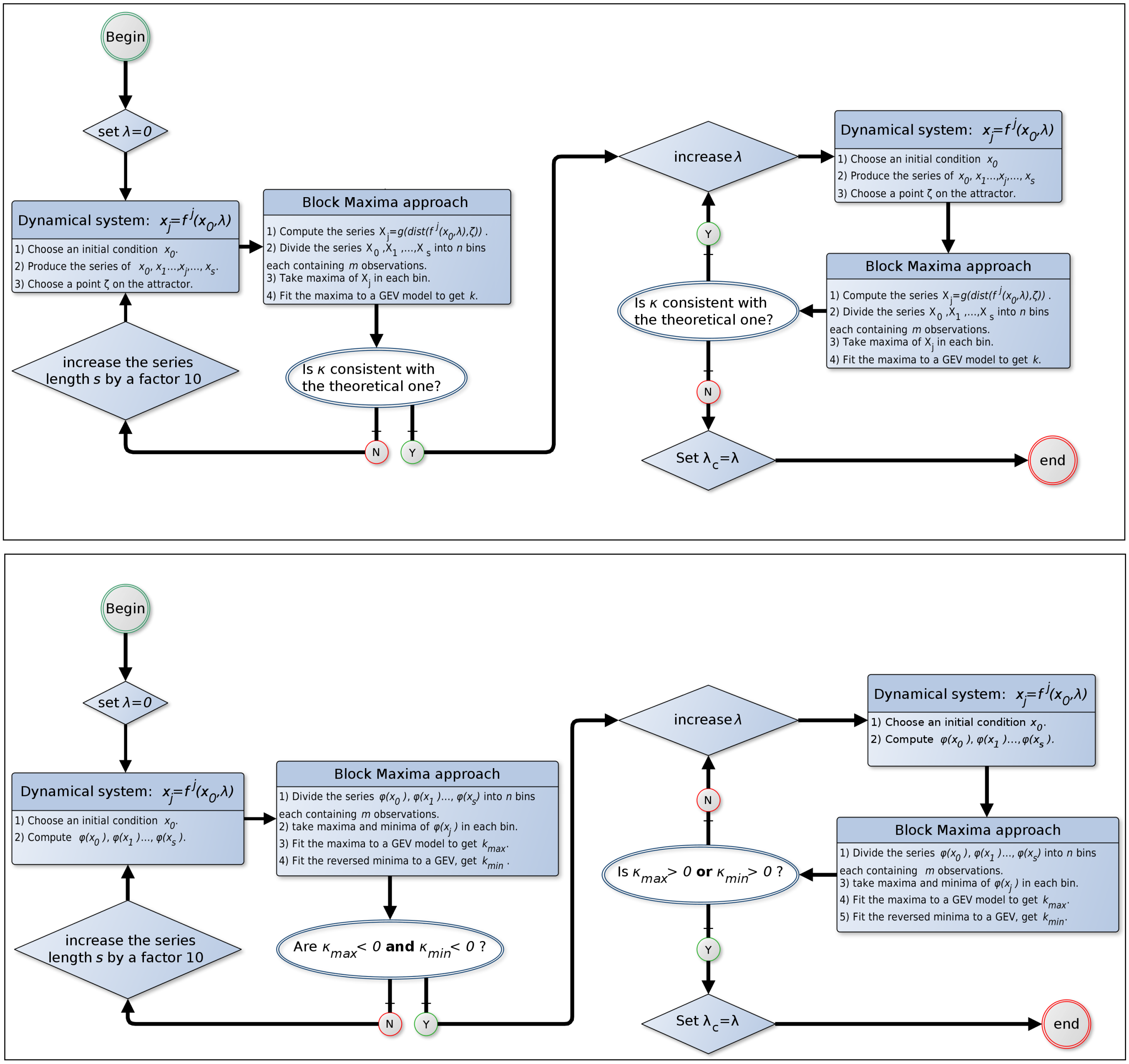

FIG. 4. Schematic representation of the numerical procedure to be used to infer the critical transition happening at $\lambda = \lambda_c$. Upper panel: dynamical indicators. Lower panel: physical indicators. See the text for further explanations